

# ON EIGENFUNCTION RESTRICTION ESTIMATES AND $L^4$ -BOUNDS FOR COMPACT SURFACES WITH NONPOSITIVE CURVATURE

CHRISTOPHER D. SOGGE AND STEVE ZELDITCH

**ABSTRACT.** If  $(M, g)$  be a two-dimensional compact boundaryless Riemannian manifold with nonpositive curvature, then we shall give improved estimates for the  $L^2$ -norms of the restrictions of eigenfunctions to unit-length geodesics, compared to the general results of Burq, Gérard and Tzvetkov [3]. By earlier results of Bourgain [2] and the first author [19], they are equivalent to improvements of the general  $L^p$ -estimates in [17] for  $n = 2$  and  $2 < p < 6$ . The proof uses the fact that the exponential map from any point in  $x_0 \in M$  is a universal covering map from  $\mathbb{R}^2 \simeq T_{x_0}M$  to  $M$  (the Cartan-Hadamard- von Mangolt theorem), which allows us to lift the necessary calculations up to the universal cover  $(\mathbb{R}^2, \tilde{g})$  where  $\tilde{g}$  is the pullback of  $g$  via the exponential map. We then prove the main estimates by using the Hadamard parametrix for the wave equation on  $(\mathbb{R}^2, \tilde{g})$  and the fact that the classical comparison theorem of Günther [6] for the volume element in spaces of nonpositive curvature gives us desirable bounds for the principal coefficient of the Hadamard parametrix, allowing us to prove our main result.

## 1. Introduction.

Let  $(M, g)$  be a compact two-dimensional Riemannian manifold without boundary. We shall assume throughout that the curvature of  $(M, g)$  is everywhere nonpositive. If  $\Delta_g$  is the Laplace-Beltrami operator associated with the metric  $g$ , then we are concerned with certain size estimates for the eigenfunctions

$$-\Delta_g e_\lambda(x) = \lambda^2 e_\lambda(x), \quad x \in M.$$

Thus we are normalizing things so that  $e_\lambda$  is an eigenfunction of the first order operator  $\sqrt{-\Delta_g}$  with eigenvalue  $\lambda$ . If  $e_\lambda$  is also normalized to have  $L^2$ -norm one, we are interested in various size estimates for the  $e_\lambda$  which are related to how concentrated they may be along geodesics. If  $\Pi$  denotes the space of all unit-length geodesics in  $M$  then our main result is the following “restriction theorem” for this problem.

**Theorem 1.1.** *Assume that  $(M, g)$  is as above. Then given  $\varepsilon > 0$  there is a  $\lambda(\varepsilon) < \infty$  so that*

$$(1.1) \quad \sup_{\gamma \in \Pi} \left( \int_\gamma |e_\lambda|^2 ds \right)^{1/2} \leq \varepsilon \lambda^{\frac{1}{4}} \|e_\lambda\|_{L^2(M)}, \quad \lambda > \lambda(\varepsilon),$$

with  $ds$  denoting arc-length measure on  $\gamma$ , and  $L^2(M)$  being the Lebesgue space with respect to the volume element  $dV_g$  for  $(M, g)$ .

Earlier, Burq, Gérard and Tzvetkov [3] showed that for *any* 2-dimensional compact boundaryless Riemannian manifold one has

$$(1.2) \quad \left( \int_\gamma |e_\lambda|^2 ds \right)^{1/2} \leq C \lambda^{\frac{1}{4}} \|e_\lambda\|_{L^2(M)},$$

with  $C$  independent of  $\gamma \in \Pi$ . The first such estimates were somewhat weaker ones of Reznikov [13] for hyperbolic surfaces, which inspired this current line of research. The estimate (1.2) is sharp for the round

---

2000 *Mathematics Subject Classification.* Primary, 35F99; Secondary 35L20, 42C99.

*Key words and phrases.* Eigenfunction estimates, negative curvature.

The authors were supported in part by the NSF. Some of this research was carried out while the first author was visiting Zhejiang University in Hangzhou, China, and he wishes to thank his colleagues there for their kindness. We also wish to thank W. Minicozzi III for many very helpful discussions.

sphere  $S^2$  because of the highest weight spherical harmonics (see [3], [19]). Burq, Gérard and Tzvetkov [3] also showed that

$$\left( \int_{\gamma} |e_{\lambda}|^4 ds \right)^{1/4} \leq C \lambda^{\frac{1}{4}} \|e_{\lambda}\|_{L^2(M)}, \quad \gamma \in \Pi,$$

and so by interpolating with this result and (1.1) one concludes that when  $M$  has nonpositive curvature  $\sup_{\gamma \in \Pi} \|e_{\lambda}\|_{L^p(\gamma)} / \|e_{\lambda}\|_{L^2(M)} = o(\lambda^{\frac{1}{4}})$  for  $2 \leq p < 4$ . An interesting but potentially difficult problem would be to show that this remains true under this hypothesis for the endpoint  $p = 4$ .

Theorem 1.1 is related to certain  $L^p$ -estimates for eigenfunctions. In [17] the first author proved that for any compact Riemannian manifold of dimension 2 one has for  $\lambda \geq 1$ ,

$$(1.3) \quad \|e_{\lambda}\|_{L^p(M)} \leq C \lambda^{\frac{1}{2}(\frac{1}{2} - \frac{1}{p})} \|e_{\lambda}\|_{L^2(M)}, \quad 2 \leq p \leq 6,$$

and

$$(1.4) \quad \|e_{\lambda}\|_{L^p(M)} \leq C \lambda^{2(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}} \|e_{\lambda}\|_{L^2(M)}, \quad 6 \leq p \leq \infty.$$

These estimates are also sharp for the round sphere  $S^2$  (see [16]). The first estimate, (1.3), is sharp because of the highest weight spherical harmonics, and thus, like (1.1) or (1.2), it measures concentration of eigenfunction mass along geodesics. The second estimate, (1.4), is sharp due to the zonal functions on  $S^2$ , which concentrate at points. The sharp variants of (1.3) and (1.4) (with different exponents) for manifolds with boundary were obtained by H. Smith and the first author in [15], and it would be interesting to obtain analogues of the results in the present paper for this setting, but this appears to be difficult.

In the last decade there have been several results showing that, for typical  $(M, g)$ , (1.4) can be improved for  $p > 6$  (see [21], [22]) to bounds of the form  $\|e_{\lambda}\|_{L^p(M)} / \|e_{\lambda}\|_{L^2(M)} = o(\lambda^{\frac{1}{2}(\frac{1}{2} - \frac{1}{p})})$  for fixed  $p > 6$ . Recently, Hassell and Tacey [9], following Bérard's [1] earlier estimate for  $p = \infty$ , showed that for fixed  $p > 6$  this ratio is  $O(\lambda^{2(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}} / \sqrt{\log \lambda})$ , which influenced the present work. Also, in [23] the authors showed that if the geodesic flow is ergodic, which is automatically the case if the curvature of  $M$  is negative, then (1.1) holds for a density one sequence of eigenfunctions.

Except for some special cases of an arithmetic nature (e.g. Zygmund [27] or Spinu [24]) there have been few cases showing that (1.3) can be improved for Lebesgue exponents with  $2 < p < 6$ . In [19], using in part results from Bourgain [2], it was shown that

$$\|e_{\lambda}\|_{L^p(M)} / \|e_{\lambda}\|_{L^2(M)} = o(\lambda^{\frac{1}{2}(\frac{1}{2} - \frac{1}{p})})$$

for some  $2 < p < 6$  if and only if

$$\sup_{\gamma \in \Pi} \|e_{\lambda}\|_{L^2(\gamma)} / \|e_{\lambda}\|_{L^2(M)} = o(\lambda^{\frac{1}{4}}).$$

Thus, we have the following corollary to Theorem 1.1.

**Corollary 1.2.** *As above, let  $(M, g)$  be a compact 2-dimensional manifold with nonpositive curvature. Then, if  $\varepsilon > 0$  and  $2 < p < 6$  are fixed there is a  $\lambda(\varepsilon, p) < \infty$  so that*

$$\|e_{\lambda}\|_{L^p(M)} \leq \varepsilon \lambda^{\frac{1}{2}(\frac{1}{2} - \frac{1}{p})} \|e_{\lambda}\|_{L^2(M)}, \quad \lambda > \lambda(\varepsilon, p).$$

We remark that an interesting open problem would be to obtain this type of result for the case of  $p = 6$ . It is valid for the standard torus  $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$  since Zygmund [27] showed that there one has  $\|e_{\lambda}\|_{L^4(\mathbb{T}^2)} / \|e_{\lambda}\|_{L^2(\mathbb{T}^2)} = O(1)$  and the classical theorem of Gauss about lattice points in the plane yields  $\|e_{\lambda}\|_{L^\infty(\mathbb{T}^2)} / \|e_{\lambda}\|_{L^2(\mathbb{T}^2)} = O(\lambda^{\frac{1}{4}})$ . Since  $p = 6$  is the exponent for which concentration at points and concentration along geodesics are both relevant, proving a general result along the lines of Corollary 1.2 would presumably have to take into account both of these phenomena. One expects, though, such a result for  $p = 6$  should be valid when  $M$  has negative curvature. This result seems to be intimately related to the problem of trying to determine when one has the endpoint improvement for the restriction problem, i.e.,  $\sup_{\gamma \in \Pi} \|e_{\lambda}\|_{L^4(\gamma)} / \|e_{\lambda}\|_{L^2(M)} = o(\lambda^{\frac{1}{4}})$ .

In [19] the first author showed that if  $\gamma_0 \in \Pi$  is not part of a periodic geodesic then

$$\|e_{\lambda}\|_{L^2(\gamma_0)} / \|e_{\lambda}\|_{L^2(M)} = o(\lambda^{\frac{1}{4}}).$$

The proof involved an estimate involving the wave equation associated with  $\Delta_g$  and a bit of microlocal (wavefront) analysis. The main step in proving Theorem 1.1 is to see that this remains valid as well if  $\gamma_0$  is part of a periodic orbit under the above curvature assumptions. We shall be able to do this by lifting the wave equation for  $(M, g)$  up to the corresponding one for its universal cover, which by a classical theorem of Hadamard [7] and von Mangolt [26], is  $(\mathbb{R}^2, \tilde{g})$ , with the metric  $\tilde{g}$  being the pullback of  $g$  via a covering map, which can be taken to be  $\exp_{x_0}$  for any  $x_0 \in M$ . By identifying solutions of wave equations for  $(M, g)$  with “periodic” ones for  $(\mathbb{R}^2, \tilde{g})$  we are able to obtain the necessary bounds using a bit of wavefront analysis and the Hadamard parametrix for  $(\mathbb{R}^2, \tilde{g})$ . Fortunately for us, by a classical volume comparison theorem of Günther [6], the leading coefficient of the Hadamard parametrix has favorable size estimates under our curvature assumptions. (It is easy to see that the contribution of the lower order terms in the Hadamard parametrix to (1.1) are straightforward to handle.)

## 2. Proof of geodesic restriction bounds.

Since the space of all unit-length geodesics is compact, in order to prove (1.1), it suffices to show that, given  $\gamma_0 \in \Pi$  and  $\varepsilon > 0$ , one can find a neighborhood  $\mathcal{N}(\gamma_0, \varepsilon)$  of  $\gamma_0$  in  $\Pi$  and a number  $\lambda(\gamma_0, \varepsilon)$  so that

$$(2.1) \quad \int_{\gamma} |e_{\lambda}|^2 ds \leq \varepsilon \lambda^{\frac{1}{2}} \|e_{\lambda}\|_{L^2(M)}^2, \quad \gamma \in \mathcal{N}(\gamma_0, \varepsilon), \quad \lambda > \lambda(\gamma_0, \varepsilon).$$

In proving this we may assume that the injectivity radius of  $(M, g)$  is ten or more. We recall also that, given  $x_0 \in M$ , the exponential map at  $x_0$ ,  $\exp_{x_0} : T_{x_0}M \simeq \mathbb{R}^2 \rightarrow M$  is a universal covering map. We shall take  $x_0$  to be the midpoint of our unit-length geodesic  $\gamma_0$ . We also shall work in geodesic polar coordinates about  $x_0$ .

If  $\tilde{g}$  is the pullback to  $\mathbb{R}^2$  of the metric  $g$  via the covering map then  $(\mathbb{R}^2, \tilde{g})$  is a Riemannian universal cover of  $(M, g)$ . Like  $(M, g)$  it also has nonpositive curvature. Additionally, rays  $t \rightarrow t(\cos \theta, \sin \theta)$ ,  $t \geq 0$ , through the origin are geodesics for  $\tilde{g}$ . Such a ray is the lift of the unit speed geodesic starting at  $x_0$ , which in our local coordinate system has the initial tangent vector  $(\cos \theta, \sin \theta)$ . Note that in these coordinates vanishing at  $x_0$ ,  $t \rightarrow t(\cos \theta, \sin \theta)$ ,  $|t| \leq 10$  are also geodesics for  $g$ . We may assume further that we have

$$(2.2) \quad \gamma_0 = \{(t, 0) : -\frac{1}{2} \leq t \leq \frac{1}{2}\}.$$

To prove (2.1) it will be convenient to fix a real-valued even function  $\chi \in \mathcal{S}(\mathcal{R})$  having the property that  $\chi(0) = 1$  and  $\hat{\chi}(t) = 0$ ,  $|t| \geq \frac{1}{4}$ , where  $\hat{\chi}$  denotes the Fourier transform of  $\chi$ . We then have that for  $T > 0$

$$\chi(T(\sqrt{-\Delta_g} - \lambda))e_{\lambda} = e_{\lambda},$$

and, therefore, to prove (2.1), it suffices to show that if  $T$  is large and fixed then there is a neighborhood<sup>1</sup>  $\mathcal{N} = \mathcal{N}(\gamma_0, T)$  of  $\gamma_0$  so that

$$(2.3) \quad \int_{\gamma} |\chi(T(\sqrt{-\Delta_g} - \lambda))f|^2 ds \leq CT^{-1}\lambda^{\frac{1}{2}}\|f\|_{L^2(M)}^2 + C'_{T,\mathcal{N}}\|f\|_{L^2(M)}^2, \quad \gamma \in \mathcal{N},$$

where  $C$  (but not  $C'_{T,\mathcal{N}}$ ) is a uniform constant depending on  $(M, g)$  but independent of  $T$  and  $\mathcal{N}$ .

To prove (2.3), we shall be able to use the wave equation as

$$(2.4) \quad \begin{aligned} \chi(T(\sqrt{-\Delta_g} - \lambda))f &= \frac{1}{2\pi T} \int_{\mathbb{R}} \hat{\chi}(t/T) e^{-it\lambda} e^{it\sqrt{-\Delta_g}} f dt \\ &= \frac{1}{\pi T} \int_{-T/4}^{T/4} \hat{\chi}(t/T) e^{-it\lambda} \cos t\sqrt{-\Delta_g} f dt + \chi(T(\sqrt{-\Delta_g} + \lambda))f, \end{aligned}$$

using the fact that  $\hat{\chi}(t)$  is even and supported in  $|t| \leq \frac{1}{4}$ . Since the kernel of the last term satisfies

$$(2.5) \quad |\partial_{x,y}^{\alpha} \chi(T(\sqrt{-\Delta_g} + \lambda))(x, y)| \leq C_{T,N} \lambda^{-N}$$

---

<sup>1</sup>We can use the topology of  $S^*M$  to define these neighborhoods, since every  $\gamma \in \Pi$  can be uniquely identified with an element  $(y, \xi) \in S^*M$ , with  $y$  being the midpoint of  $\gamma$  and  $\xi$  being the direction of  $\gamma$  at  $y$ .

for any  $N$  in compact subsets of any local coordinate system, to prove (2.3) it suffices to show that

$$(2.6) \quad \int_{\gamma} \left| \frac{1}{\pi T} \int_{-T/4}^{T/4} \hat{\chi}(t/T) e^{-it\lambda} \cos t \sqrt{-\Delta_g} f dt \right|^2 ds \leq (CT^{-1} \lambda^{\frac{1}{2}} + C'_{T,N}) \|f\|_{L^2(M)}, \quad \gamma \in \mathcal{N}(\gamma_0, T).$$

If  $\gamma_0$  is not part of a periodic geodesic of period  $\leq T$ , then we can easily prove (2.6) just by using wavefront analysis and arguments that are similar to the proof of the Duistermaat-Guillemin theorem [5]. This was done in [19], but we shall repeat the argument here for the sake of completeness and since it motivates what is needed to handle the argument when  $\gamma_0$  is a portion of a periodic geodesic of period  $\leq T$ .

To handle the latter case we shall exploit the relationship between solutions of the wave equation on  $(M, g)$  of the form

$$(2.7) \quad \begin{cases} (\partial_t^2 - \Delta_g)u(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times M \\ u(0, \cdot) = f, \quad \partial_t u(0, \cdot) = 0, \end{cases}$$

and certain ones on  $(\mathbb{R}, \tilde{g})$

$$(2.8) \quad \begin{cases} (\partial_t^2 - \Delta_{\tilde{g}})\tilde{u}(t, \tilde{x}), & (t, \tilde{x}) \in \mathbb{R}_+ \times \mathbb{R}^2 \\ \tilde{u}(0, \cdot) = \tilde{f}, \quad \partial_t \tilde{u}(0, \cdot) = 0. \end{cases}$$

Note that  $u(t, x) = (\cos(t\sqrt{-\Delta_g})f)(x)$  is the solution of (2.7).

To describe the relationship between the two equations we shall use the deck transformations associated with our universal covering map

$$(2.9) \quad p = \exp_{x_0} : \mathbb{R}^2 \rightarrow M.$$

Recall that an automorphism for  $(\mathbb{R}^2, \tilde{g})$ ,  $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , is a deck transformation if

$$p \circ \alpha = p.$$

In this case we shall write  $\alpha \in \text{Aut}(p)$ . In the case where  $\mathbb{T}^2$  is the standard two-torus, each  $\alpha$  would just be translation in  $\mathbb{R}^2$  with respect to some  $j \in \mathbb{Z}^2$ . Motivated by this if  $\tilde{x} \in \mathbb{R}^2$  and  $\alpha \in \text{Aut}(p)$ , let us call  $\alpha(\tilde{x})$  the translate of  $\tilde{x}$  by  $\alpha$ . then we recall a set  $D \subset \mathbb{R}^2$  is called a fundamental domain of our universal covering  $p$  if every point in  $\mathbb{R}^2$  is the translate of exactly one point in  $D$ . Of course there are infinitely many fundamental domains, but we may assume that ours is relatively compact, connected and contains the ball of radius 2 centered at the origin in view of our assumption about the injectivity radius of  $(M, g)$ . We can then think of our unit geodesic  $\gamma_0 = \{(t, 0) : |t| \leq \frac{1}{2}\}$  (written in geodesic polar coordinates as above) both as one in  $(M, g)$  and one in the fundamental domain which is of the same form. Likewise, a function  $f(x)$  on  $M$  is uniquely identified by one  $f_D(\tilde{x})$  on  $D$  if we set  $f_D(\tilde{x}) = f(x)$ , where  $\tilde{x}$  is the unique point in  $D \cap p^{-1}(x)$ . Using  $f_D$  we can define a “periodic extension”,  $\tilde{f}$ , of  $f$  to  $\mathbb{R}^2$  by defining  $\tilde{f}(\tilde{y})$  to be equal to  $f_D(\tilde{x})$  if  $\tilde{x} = \tilde{y}$  modulo  $\text{Aut}(p)$ , i.e. if  $(\tilde{x}, \alpha) \in D \times \text{Aut}(p)$  are the unique pair so that  $\tilde{y} = \alpha(\tilde{x})$ . Note then that  $\tilde{f}$  is periodic with respect to  $\text{Aut}(p)$  since we necessarily have that  $\tilde{f}(\tilde{x}) = \tilde{f}(\alpha(\tilde{x}))$  for every  $\alpha \in \text{Aut}(p)$ .

We can now describe the relationship between the wave equations (2.7) and (2.8). First, if  $(f(x), 0)$  is the Cauchy data in (2.7) and  $(\tilde{f}(\tilde{x}), 0)$  is the periodic extension to  $(\mathbb{R}^2, \tilde{g})$ , then the solution  $\tilde{u}(t, \tilde{x})$  to (2.8) must also be a periodic function of  $\tilde{x}$  since  $\tilde{g}$  is the pullback of  $g$  via  $p$  and  $p = p \circ \alpha$ . As a result, we have that the solution to (2.7) must satisfy  $u(t, x) = \tilde{u}(t, \tilde{x})$  if  $\tilde{x} \in D$  and  $p(\tilde{x}) = x$ . Another way of saying this is that if  $\tilde{f}$  is the pullback of  $f$  via  $p$  and  $t$  is fixed then  $\tilde{u}(t, \cdot)$  solving (2.8) must be the pullback of  $u(t, \cdot)$  in (2.7). Thus, periodic solutions to (2.8) correspond uniquely to solutions of (2.7). In other words, we have the important formula for the wave kernels

$$(2.10) \quad (\cos(t\sqrt{-\Delta_g})(x, y) = \sum_{\alpha \in \text{Aut}(p)} (\cos(t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x}, \alpha(\tilde{y})),$$

if  $\tilde{x}$  and  $\tilde{y}$  are the unique points in  $D$  for which  $p(\tilde{x}) = x$  and  $p(\tilde{y}) = y$ .

Note that the sum in (2.10) only has finitely many nonzero terms for a given  $(x, y, t)$  since, by the finite propagation speed for  $\square_{\tilde{g}} = \partial_t^2 - \Delta_{\tilde{g}}$ , the summands in the in the right all vanish when  $d_{\tilde{g}}(\tilde{x}, \alpha(\tilde{y})) > t$ . For instance, if  $x = y = x_0$  the number of nontrivial terms would equal the cardinality of  $p^{-1}(x_0) \cap \{\tilde{x} \in \mathbb{R}^2 : |\tilde{x}| \leq t\}$  where  $|\tilde{x}|$  denotes the Euclidean length, due to the fact that  $d_{\tilde{g}}(0, \tilde{x}) = |\tilde{x}|$ . Despite this, the number

of nontrivial terms will grow exponentially in  $t$  if the curvature is bounded from above by a fixed negative constant.

To see this, let us review one last thing before focusing more closely on the proof of our restriction-estimate. As we shall see, even though there can be an exponentially growing number of nontrivial terms in the right hand side of (2.10), which could create havoc for our proofs if we are not careful, this turns out to be related to something that will actually be beneficial for our calculations.

These facts are related to the fact that in the geodesic polar coordinates we are using,  $(t \cos \theta, t \sin \theta)$ ,  $t > 0$ ,  $\theta \in (-\pi, \pi]$ , for  $(\mathbb{R}^2, \tilde{g})$ , the metric  $\tilde{g}$  takes the form

$$(2.11) \quad ds^2 = dt^2 + \mathcal{A}^2(t, \xi) d\theta^2,$$

where we may assume that  $\mathcal{A}(t, \theta) > 0$  for  $t > 0$ . Consequently, the volume element in these coordinates is given by

$$(2.12) \quad dV_g(t, \theta) = \mathcal{A}(t, \theta) dt d\theta,$$

and by Günther's [6] comparison theorem if the curvature of  $(M, g)$  and hence that of  $(\mathbb{R}^2, \tilde{g})$  is nonpositive, we have

$$(2.13) \quad \mathcal{A}(t, \theta) \geq t.$$

Furthermore, if one assumes that the curvature is  $\leq -\kappa^2$ , with  $\kappa > 0$  then one has

$$(2.14) \quad \mathcal{A}(t, \theta) \geq \frac{1}{\kappa} \sinh(\kappa t).$$

Since the volume element for two-dimensional Euclidean space in polar coordinates is  $t dt d\theta$  and that of the hyperbolic plane with constant curvature  $-\kappa^2$  is  $\frac{1}{\kappa} \sinh(\kappa t) dt d\theta$ , Günther's volume comparison theorem says that in geodesic polar coordinates the volume element for spaces of nonpositive curvature is at least that of  $\mathbb{R}^2$  with the flat metric, while if the curvature is bounded above by  $-\kappa^2$  the volume element is at least that of the hyperbolic plane of constant curvature  $-\kappa^2$ . In the latter case, as we warned, the number of nontrivial terms in the sum in the right side of (2.10) will be at least bounded below by a multiple of  $e^{\kappa t}$  as  $t \rightarrow +\infty$ .

Let us now turn to the proof of (2.6) and hence Theorem 1.1. Given  $\gamma \in \Pi$  we let  $T^*\gamma \subset T^*M$  and  $S^*\gamma \subset S^*M$  be the cotangent and unit cotangent bundles over  $\gamma$ , respectively. Thus, if  $(x, \xi) \in T^*\gamma$  then  $\xi_\#$  is a tangent vector to  $\gamma$  at  $x$  if  $T^*M \ni \xi \rightarrow \xi_\# \in TM$  is the standard musical isomorphism, which, in local coordinates, sends  $\xi = (\xi_1, \xi_2) \in T_x^*M$  to  $\xi_\# = (\xi_\#^1, \xi_\#^2)$  with  $\xi_\#^j = \sum_k g^{jk}(x) \xi_k$ . Then if  $\Phi_t : S^*M \rightarrow S^*M$  denotes geodesic flow in the unit cotangent bundle over  $M$ , and  $(x, \xi) \in S^*\gamma$  we let  $L(x, \xi)$  be the minimal  $t > 0$  so that  $\Phi_t(x, \xi) = (x, \xi)$  and define it to be  $+\infty$  if no such time  $t$  exists. Then if  $\gamma$  is not part of a periodic geodesic this quantity is  $+\infty$  on  $S^*\gamma$ , and if it is then it is constant on  $S^*\gamma$  and equal to the minimal period of the geodesic,  $\ell(\gamma)$  (which must be larger than ten because of our assumptions). Note also that  $L(x, \xi)$  can also be thought of as a function on  $S^*M$ , and that, in this case, it is lower semicontinuous.

Recall that we are working in geodesic polar coordinates vanishing at  $x_0$ , the midpoint of  $\gamma_0$ , and that  $\gamma_0$  is of the form (2.2) in these coordinates. Let us choose  $\beta \in C_0^\infty(\mathbb{R})$  equal to one on  $[-\frac{3}{4}, \frac{3}{4}]$  but 0 outside  $[-1, 1]$ . We then let  $b_\varepsilon(x, D)$  and  $B_\varepsilon(x, D)$  be zero-order pseudodifferential operators which in the above local coordinates have symbols

$$b_\varepsilon(x, \xi) = \beta(|x|) \beta(\xi_2/\varepsilon|\xi|), \quad \text{and} \quad B_\varepsilon(x, \xi) = \beta(|x|)(1 - \beta(\xi_2/\varepsilon|\xi|)),$$

respectively.

Our first claim is that if  $\varepsilon > 0$  and  $\gamma \in \Pi$  are fixed, then we can find a neighborhood  $\mathcal{N}(\gamma_0, \varepsilon)$  of  $\gamma_0$  so that

$$(2.15) \quad \int_{-T/4}^{T/4} \int_\gamma \left| B_\varepsilon \circ \cos(t\sqrt{-\Delta_g}) f \right|^2 ds dt \leq C_{T, \varepsilon} \|f\|_{L^2(M)}^2, \quad \gamma \in \mathcal{N}(\gamma_0, \varepsilon),$$

which, by an application of the Schwartz inequality, would yield part of (2.6), namely,

$$(2.16) \quad \int_\gamma \left| \frac{1}{\pi T} \int_{-T/4}^{T/4} \hat{\chi}(t/T) e^{-i\lambda t} B_\varepsilon \circ \cos(t\sqrt{-\Delta_g}) f dt \right|^2 ds \leq C'_{T, \varepsilon} \|f\|_{L^2(M)}^2, \quad \gamma \in \mathcal{N}(\gamma_0, \varepsilon).$$

If  $R_\gamma$  denotes the restriction to  $\gamma \in \Pi$ , then (2.15) follows from the fact that the operator

$$f \rightarrow R_\gamma(A \circ \cos(t\sqrt{-\Delta_g})f),$$

regarded as an operator from  $C^\infty(M) \rightarrow C^\infty(\gamma \times [-T/4, T/4])$ , is a Fourier integral operator of order zero which is locally a canonical graph (i.e., nondegenerate) if  $\text{supp } A(x, \xi) \cap S^*\gamma = \emptyset$ , and hence a bounded operator from  $L^2(M)$  to  $L^2(\gamma \times [-T/4, T/4])$ . since  $B_\varepsilon(x, \xi)$  vanishes on a neighborhood of  $S^*\gamma_0$ , we conclude that this is the case  $A = B_\varepsilon$  for  $\gamma \in \Pi$  close to  $\gamma_0$ , which gives us (2.15). The  $L^2$ -boundedness of nondegenerate Fourier integrals is a theorem of Hörmander [10], while the observation about  $R_\gamma(A \circ \cos(t\sqrt{-\Delta_g}))$  is one of Tataru [25]. It is also easy to check the latter, because, for fixed  $t$ ,  $e^{it\sqrt{-\Delta_g}} : C^\infty(M) \rightarrow C^\infty(M)$  is a nondegenerate Fourier integral operator, and, therefore, one needs only to verify the assertion when  $t = 0$ , in which case it is an easy calculation using any parametrix for the half-wave operator.

The estimate (2.16) holds for any  $\gamma_0 \in \Pi$ . Let us now argue that if  $\ell(\gamma_0)$ , the period of  $\gamma_0$ , is larger than  $T$  or if  $\gamma_0$  is not part of a periodic geodesic, then we have also have favorable bounds if  $B_\varepsilon$  is replaced by  $b_\varepsilon$ , with  $\varepsilon > 0$  sufficiently small. To do this, we recall that the wave front set of the kernel of  $b_\varepsilon \circ \cos(t\sqrt{-\Delta_g}) \circ b_\varepsilon^*$  is contained in

$$(2.17) \quad \{(x, t, \xi, \tau; y, -\eta) : \Phi_{\pm t}(x, \xi) = (y, \eta), \tau^2 = \sum g^{jk}(x)\xi_j\xi_k, (x, \xi), (y, \eta) \in \text{supp } b_\varepsilon\}.$$

To exploit this, let  $W_\gamma$  be the operator

$$(2.18) \quad W_\gamma f = R_\gamma \left( \frac{1}{\pi T} \int_{-T/4}^{T/4} \hat{\chi}(t/T) e^{-i\lambda t} b_\varepsilon \circ \cos(t\sqrt{-\Delta_g}) f dt \right).$$

Our goal then is to show, that under the present assumption that  $\ell(\gamma_0) > T$ ,

$$\|W_\gamma\|_{L^2(M) \rightarrow L^2(\gamma)} \leq CT^{-\frac{1}{2}} \lambda^{\frac{1}{4}} + C_{T, b_\varepsilon}$$

for  $\gamma \in \Pi$  belonging to some neighborhood  $\mathcal{N}(\gamma_0, T, \varepsilon)$  of  $\gamma_0$ . This is equivalent to showing that the dual operator  $W_\gamma^* : L^2(\gamma) \rightarrow L^2(M)$  with the same norm, and since

$$\|W_\gamma^* g\|_{L^2(M)}^2 = \int_\gamma W_\gamma W_\gamma^* g \bar{g} ds \leq \|W_\gamma W_\gamma^* g\|_{L^2(\gamma)} \|g\|_{L^2(\gamma)},$$

we would be done if we could show that

$$(2.19) \quad \|W_\gamma W_\gamma^* g\|_{L^2(\gamma)} \leq \left( CT^{-1} \lambda^{\frac{1}{2}} + C_{T, b_\varepsilon} \right) \|g\|_{L^2(\gamma)}.$$

But, by Euler's formula, the kernel of  $4W_\gamma W_\gamma^*$  is  $K|_{\gamma \times \gamma}$ , where  $K(x, y)$ ,  $x, y \in M$  is the kernel of the operator

$$b_\varepsilon \circ \rho(T(\sqrt{-\Delta_g} - \lambda)) \circ b_\varepsilon^* + b_\varepsilon \circ \rho(T(\sqrt{-\Delta_g} + \lambda)) \circ b_\varepsilon^* + 2b_\varepsilon \circ \chi(T(\sqrt{-\Delta_g} - \lambda)) \chi(T(\sqrt{-\Delta_g} + \lambda)) \circ b_\varepsilon^*,$$

if  $\rho(\tau) = (\chi(\tau))^2$ . The last two terms satisfy bounds like those in (2.5) (with constant depending on  $T$  and  $b_\varepsilon$ ), and the first term is

$$(2.20) \quad \frac{1}{\pi T} \int_{-T/2}^{T/2} \hat{\rho}(t/T) e^{-i\lambda t} (b_\varepsilon \circ \cos(t\sqrt{-\Delta_g}) \circ b_\varepsilon^*)(x, y) dt.$$

We are using the fact that  $\hat{\rho} = \hat{\chi} * \hat{\chi}$  is supported in  $[-\frac{1}{2}, \frac{1}{2}]$ . In view of (2.17), if  $\varepsilon > 0$  is sufficiently small, since we are assuming that  $\ell(\gamma_0) > T$ , it follows that we can find a neighborhood  $\mathcal{N}$  of  $\gamma_0$  in  $M$  so that  $(b_\varepsilon \circ \cos(t\sqrt{-\Delta_g}) \circ b_\varepsilon^*)(x, y)$  is smooth on  $\mathcal{N} \times \mathcal{N}$  when  $t \geq 2$ . Thus, on  $\mathcal{N} \times \mathcal{N}$  the difference between (2.19) and

$$K(x, y) = \frac{1}{\pi T} \int_{-T/2}^{T/2} \beta(t/5) \rho(t/T) e^{-i\lambda t} (b_\varepsilon \circ \cos(t\sqrt{-\Delta_g}) \circ b_\varepsilon^*)(x, y) dt$$

is  $O_{T, b_\varepsilon}(1)$ . But, by using the Hadamard parametrix (see below) one finds that

$$(2.21) \quad |K(x, y)| \leq CT^{-1} \lambda^{\frac{1}{2}} (d_g(x, y))^{-\frac{1}{2}} + C_{b_\varepsilon, T} (1 + \lambda(1 + \lambda d_g(x, y))^{-\frac{3}{2}}), \quad x, y \in \mathcal{N},$$

for some uniform constant  $C$ , which is independent of  $\varepsilon$ ,  $T$  and  $\lambda$ . Since, by Young's inequality, the integral operator with kernel  $K|_{\gamma \times \gamma}$  is bounded from  $L^2(\gamma) \rightarrow L^2(\gamma)$  with norm bounded by  $CT^{-1} \lambda^{\frac{1}{2}} + C_{b_\varepsilon, T}$  if  $\gamma \subset \mathcal{N}$ , we get (2.19), which finishes the proof that (2.3) holds provided that  $\ell(\gamma_0) > T$ .

The above argument used the fact that if  $\ell(\gamma_0) > T$ , with  $T$  fixed, then if  $\varepsilon > 0$  is small enough and  $(x, \xi) \in \text{supp } b_\varepsilon$  with  $x \in \gamma_0$  then  $\Phi_t(x, \xi) \notin \text{supp } b_\varepsilon$  for  $2 < |t| \leq T/2$ . In effect, this allowed us to cut the effect of loops through  $\gamma_0$  of its extension of length  $T$  from our main calculation, since they were all transverse. If  $\gamma_0 \in \Pi$  is part of a periodic geodesic of period  $\leq T$ , i.e.,  $\ell(\gamma_0) \leq T$ , then this need not be true. On the other hand, if  $T$  is fixed and  $(x, \xi)$  is as above, then for sufficiently small  $\varepsilon$  we will have

$$(2.22) \quad \Phi_{\pm t}(x, \xi) \notin \text{supp } b_\varepsilon, \text{ if } x \in \gamma_0, \text{ and } t \notin \bigcup_{j \in \mathbb{Z}} [j\ell(\gamma_0) - 2, j\ell(\gamma_0) + 2].$$

Note that our assumption that the injectivity radius of  $(M, g)$  is 10 or more implies that

$$\ell(\gamma_0) \geq 10.$$

To exploit this, we shall use (2.10) which relates the wave kernel for  $(M, g)$  with the one for its universal cover using the covering map given by  $p = \exp_{x_0}$  with  $x_0$  being the midpoint of  $\gamma_0$ . Note that the points  $\alpha(0)$ ,  $\alpha \in \text{Aut}(p)$  exactly correspond to geodesic loops through  $x_0$ , with looping time being equal to the distance from  $\alpha(0)$  to the origin in  $\mathbb{R}^2$ . Just a few of these correspond to smooth loops through  $x_0$  along the periodic geodesic containing  $\gamma_0$ . Since we are assuming that we are working with local coordinates on  $(M, g)$  and global geodesic polar ones on  $(\mathbb{R}^2, \tilde{g})$  so that  $\gamma_0$  is of the form (2.2), the automorphisms with this property are exactly the  $\alpha_j \in \text{Aut}(p)$ ,  $j \in \mathbb{Z}$  for which

$$(2.23) \quad \alpha_j(0) = (j\ell(\gamma_0), 0).$$

Note that  $G_{\gamma_0} = \{\alpha_j\}_{j \in \mathbb{Z}}$  is a cyclic subgroup of  $\text{Aut}(p)$  with generator  $\alpha_1$ , which is the stabilizer group for the lift of periodic geodesic containing  $\gamma_0$ . Consequently, we can choose  $\varepsilon > 0$  small enough and a neighborhood  $\mathcal{N}$  of  $\gamma_0$  in  $M$  so that<sup>2</sup>

$$(2.24) \quad (b_\varepsilon \circ \cos(t\sqrt{-\Delta_{\tilde{g}}}) \circ b_\varepsilon^*)(\tilde{x}, \alpha(\tilde{y})) \in C^\infty(\mathcal{N} \times \mathcal{N} \times [j\ell(\gamma_0) - 2, j\ell(\gamma_0) + 2]), \quad \text{if } \text{Aut}(p) \ni \alpha \notin G_{\gamma_0}.$$

Therefore, by (2.22)–(2.24), if we repeat the arguments that were used to prove (2.19), we conclude that we would have

$$(2.25) \quad \int_{\gamma} \left| \frac{1}{\pi T} \int_{-\infty}^{\infty} \hat{\chi}(t/T) e^{-i\lambda t} b_\varepsilon \circ \cos(t\sqrt{-\Delta_g}) f dt \right|^2 ds \leq (CT^{-\frac{1}{4}} \lambda^{\frac{1}{4}} + C_{T, b_\varepsilon})^2 \|f\|_{L^2(M)}^2, \quad \gamma \in \mathcal{N}(\gamma_0, T),$$

for some neighborhood  $\mathcal{N}(\gamma_0, T)$  in  $\Pi$ , if we could show that if the  $\alpha_j$  are as in (2.23) and

$$(2.26) \quad K(x, y) = \frac{1}{\pi T} \sum_{\{j \in \mathbb{Z}_+ : jt(\gamma_0) \leq T/2\}} \int_{-\infty}^{\infty} \beta((s - j\ell(\gamma_0))/5) \hat{\rho}(s/T) e^{-is\lambda} \\ \times (b_\varepsilon \circ \cos s\sqrt{-\Delta_{\tilde{g}}} \circ b_\varepsilon^*)(\tilde{x}, \alpha_j(\tilde{y})) ds,$$

then

$$(2.27) \quad |K(x, y)| \leq (CT^{-1} \lambda^{\frac{1}{2}} (d_g(x, y))^{-\frac{1}{2}} + T^{-\frac{1}{2}} \lambda^{\frac{1}{2}} + C_{T, b_\varepsilon} (1 + \lambda (1 + \lambda d_g(x, y))^{-\frac{3}{2}})), \quad x, y \in \mathcal{N},$$

with  $\mathcal{N}$  being some neighborhood in  $M$  of  $\gamma_0$  (depending on  $T$ ). The second term in the right side of this inequality did not occur in the previous steps. It comes from the terms in (2.26) with  $j \neq 0$ . Also, the fact that (2.27) yields (2.25) just follows from an application of Young's inequality.

To prove (2.27), it suffices to see that we can find  $\mathcal{N}$  as above so that

$$(2.28) \quad \left| \int \beta((s - j\ell(\gamma_0))/5) \hat{\rho}(s/T) e^{-is\lambda} (b_\varepsilon \circ \cos s\sqrt{-\Delta_{\tilde{g}}} \circ b_\varepsilon^*)(\tilde{x}, \alpha_j(\tilde{y})) ds \right| \\ \leq C \lambda^{\frac{1}{2}} \left( \max\{d_g(\tilde{x}, \alpha_j(\tilde{y})), e^{\kappa d_g(\tilde{x}, \alpha_j(\tilde{y}))}\} \right)^{-\frac{1}{2}} + C_{T, b_\varepsilon}, \quad x, y \in \mathcal{N}, \quad 0 \neq |j|\ell(\gamma_0) \leq T,$$

<sup>2</sup>We should point out that we are abusing notation a bit in (2.24). The last factor denotes the kernel of the integral operator on  $M$  with kernel  $H(x, y) = (\cos t\sqrt{-\Delta_{\tilde{g}}})(\tilde{x}, \alpha(\tilde{y}))$  is composed on the left and right by  $b_\varepsilon$  and  $b_\varepsilon^*$ , respectively, and as before we are identifying points  $x$  in  $M$  with their cousin  $\tilde{x}$  in the fundamental domain. In the coordinate systems we are using, though, both are the same when we are close to  $\gamma_0$ .

assuming that the curvature of  $(M, g)$  is everywhere  $\leq -\kappa^2$ ,  $\kappa \geq 0$ , while for  $j = 0$ , we have

$$(2.29) \quad \left| \int \beta(s/5) \hat{\rho}(s/T) e^{-is\lambda} (b_\varepsilon \circ \cos s \sqrt{-\Delta_g} \circ b_\varepsilon^*)(x, y) ds \right| \\ \leq C \lambda^{\frac{1}{2}} (d_g(x, y))^{-\frac{1}{2}} + C_{T, b_\varepsilon} (1 + \lambda (1 + \lambda d_g(x, y))^{-\frac{3}{2}}), \quad x, y \in \mathcal{N}.$$

Note that  $d_{\tilde{g}}(\tilde{x}, \alpha_j(\tilde{y})) \in [j\ell(\gamma_0) - 1, j\ell(\gamma_0) + 1]$  when  $x, y \in \gamma_0$  and hence  $d_{\tilde{g}}(\tilde{x}, \alpha_j(\tilde{y})) \geq |j|$  when  $x, y \in \mathcal{N}$  with  $\mathcal{N}$  being a small neighborhood of  $\gamma_0$  in  $M$ . We shall assume that this is the case in what follows. We then get (2.27) by summing over  $j$ . (Observe that if the curvature is assumed to be bounded below by a negative constant, we get something a bit stronger than (2.27) where in the second term we may replace  $T^{-\frac{1}{2}}$  by  $T^{-1}$ .)

Both (2.28) and (2.29) are routine consequences of stationary phase and the Hadamard parametrix for the wave equation.

To prove (2.29) let  $\phi(x, y)$  denote geodesic normal coordinates of  $y$  about  $x$ . Then if  $|t| \leq 5$ , by the Hadamard parametrix (see [11] or [20]) and the composition calculus for Fourier integral operators (see Chapter 6 in [18])

$$(2.30) \quad (b_\varepsilon \circ \cos(t\sqrt{-\Delta_g}) \circ b_\varepsilon^*)(x, y) = \sum_{\pm} \int_{\mathbb{R}^2} e^{i\phi(x, y) \cdot \xi \pm it|\xi|} a_\varepsilon(x, y, \xi) d\xi + O_\varepsilon(1),$$

where  $a_\varepsilon \in S_{1,0}^0$  depends on  $-\Delta_g$  and  $b_\varepsilon$  but satisfies

$$(2.31) \quad |a_\varepsilon| \leq C, \quad \text{and} \quad |\partial_{x,y}^\alpha \partial_\xi^\sigma a_\varepsilon| \leq C_{\varepsilon\alpha\sigma} (1 + |\xi|)^{-|\sigma|}.$$

The first constant is independent of  $C$  and only depends on the size of the symbol of  $b_\varepsilon$ , which is  $\leq \|\beta\|_{L^\infty(\mathbb{R})}^4$ . Recall (see [18]) the following fact about the Fourier transform of a density times Lebesgue measure on the circle  $S^1 = \{\Theta = (\cos \theta, \sin \theta)\}$ ,

$$(2.32) \quad \int_0^{2\pi} e^{iw \cdot \Theta} a_\varepsilon(x, y, \Theta) d\theta = |2\pi w|^{-\frac{1}{2}} \sum_{\pm} e^{\pm i|w|} a_\varepsilon(x, y, \pm w) + O_\varepsilon(|w|^{-\frac{3}{2}}), \quad |w| \geq 1,$$

where the constants for the last term depend on the size of finitely many constants in (2.32). Since  $|\phi(x, y)| = d_g(x, y)$ , if we combine (2.30) and (2.31), we find that, modulo a  $O_\varepsilon(1)$  term, if  $\psi(s) = \beta(s/5)\hat{\rho}(s/T)$ , then when  $d_g(x, y) \geq \lambda^{-1}$ , the quantity in (2.29) is the sum over  $\pm$  of a fixed multiple of

$$(d_g(x, y))^{-\frac{1}{2}} \int_0^\infty (\hat{\psi}(\lambda - r) + \hat{\psi}(\lambda + r)) e^{\pm i r d_g(x, y)} a_\varepsilon(x, y, \pm r \phi(x, y)) r^{\frac{1}{2}} dr \\ + O_\varepsilon((d_g(x, y))^{-\frac{3}{2}} \int_0^\infty (|\hat{\psi}(\lambda - r)| + |\hat{\psi}(\lambda + r)|) (1 + r)^{-\frac{1}{2}} dr).$$

By (2.31), the first term in is  $O(\|a_\varepsilon\|_\infty (\lambda d_g(x, y))^{-\frac{1}{2}})$ , since  $|\hat{\psi}(\tau)| \leq C_N(1 + |\tau|)^{-N}$  for any  $N$ . Since the last term is  $O_\varepsilon(\lambda^{-1/2} (d_g(x, y))^{-\frac{3}{2}})$ , we have established (2.29) when  $d_g(x, y) \geq \lambda^{-1}$ . The fact that it is also  $O(\lambda) + O_\varepsilon(1)$  is a simple consequence of (2.30) and (2.31) which gives the bounds for  $d_g(x, y) \leq \lambda^{-1}$  and concludes the proof of (2.29).

To prove (2.29) we can exploit the fact that, unlike the case of  $t = 0$ , if  $t \neq 0$  then  $\cos t\sqrt{-\Delta_g} : C^\infty(M) \rightarrow C^\infty(M)$  is a conormal Fourier integral operator with singular support of codimension one. Based on this and (2.17) we deduce that if  $(x, t, \xi, \tau; y, \eta)$  is in the wave front set of

$$(\cos(t\sqrt{-\Delta_g}))(\tilde{x}, \alpha_j(\tilde{y})), \quad j \neq 0,$$

and both  $x$  and  $y$  are on  $\gamma_0$  then both  $\xi$  and  $\eta$  must be on the first coordinate axis. Therefore, since the symbol,  $b_\varepsilon(x, \xi)$ , of  $b_\varepsilon$  equals one when  $x \in \gamma_0$  and  $\xi$  is in a conic neighborhood of this axis (depending on  $\varepsilon$ ), we conclude that there must be a neighborhood  $\mathcal{N}$  of  $\gamma_0$  in  $M$  so that

$$(b_\varepsilon \circ \cos(t\sqrt{-\Delta_g}) \circ b_\varepsilon^*)(\tilde{x}, \alpha_j(\tilde{y})) - (\cos t\sqrt{-\Delta_g})(\tilde{x}, \alpha_j(\tilde{y})) \in C^\infty(\mathcal{N} \times \mathcal{N}), \quad 0 \neq |j|\ell(\gamma_0) \leq T.$$



Because of this, we would have the remaining inequality, (2.28), if we could show that

$$(2.33) \quad \left| \int \beta((s - j\ell(\gamma_0))/5) \hat{\rho}(s/T) e^{-is\lambda} (\cos s \sqrt{-\Delta_{\tilde{g}}}) (\tilde{x}, \alpha_j(\tilde{y})) ds \right| \\ \leq C\lambda^{\frac{1}{2}} \left( \max\{d_g(\tilde{x}, \alpha_j(\tilde{y})), e^{\kappa d_g(\tilde{x}, \alpha_j(\tilde{y}))}\} \right)^{-\frac{1}{2}} + C_T \quad x, y \in \mathcal{N}, \quad 0 \neq |j|\ell(\gamma_0) \leq T.$$

To prove this, we shall use the fact that on  $(\mathbb{R}^2, \tilde{g})$  we can use the Hadamard parametrix even for large times. Recall that the Hadamard parametrix says that if we set

$$\mathcal{E}_0(t, x) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix \cdot \xi} \cos(t|\xi|) d\xi,$$

and define  $\mathcal{E}_\nu$ ,  $\nu = 1, 2, 3, \dots$  recursively by  $2\mathcal{E}_\nu(t, x) = t \int_0^t \mathcal{E}_{\nu-1}(s, x) ds$ ,  $\nu = 1, 2, 3, \dots$ , then there are functions  $w_\nu \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$  so that we have

$$(\cos(t\sqrt{-\Delta_{\tilde{g}}})(x, y) = \sum_{\nu=0}^N w_\nu(x, y) \mathcal{E}_\nu(t, d_{\tilde{g}}(x, y)) + R_N(t, x, y),$$

where for  $n = 2$ ,  $R_N \in L_{loc}^\infty(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2)$  if  $N \geq 10$ . We are abusing the notation a bit by putting  $\mathcal{E}_\nu(t, r)$  equal to the radial function  $\mathcal{E}_\nu(t, x)$  for some  $|x| = r$ . The  $\mathcal{E}_\nu$ ,  $\nu = 1, 2, 3, \dots$ , are Fourier integrals of order  $-\nu$ ; for instance,

$$\mathcal{E}_1(t, x) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix \cdot \xi} \frac{t \sin t|\xi|}{2|\xi|} d\xi.$$

As a result of this, we would have (2.33) if we could show that

$$(2.34) \quad \left| w_0(\tilde{x}, \alpha_j(\tilde{y})) \iint \beta((s - j\ell(\gamma_0))/5) \hat{\rho}(s/T) e^{-i\lambda s} e^{i(\tilde{x} - \alpha_j(\tilde{y})) \cdot \xi} \cos(s|\xi|) d\xi ds \right| \\ \leq C\lambda^{\frac{1}{2}} \left( \max\{d_g(\tilde{x}, \alpha_j(\tilde{y})), e^{\kappa d_g(\tilde{x}, \alpha_j(\tilde{y}))}\} \right)^{-\frac{1}{2}}, \quad j = 1, 2, \dots,$$

as well as

$$(2.35) \quad \left| \int \beta((s - j\ell(\gamma_0))/5) \hat{\rho}(s/T) e^{-is\lambda} \mathcal{E}_\nu(s, d_g(\tilde{x}, \alpha_j(\tilde{y}))) ds \right| \leq C_\nu, \quad 0 \neq j\ell(\gamma_0) \leq T, \quad \nu = 1, 2, 3, \dots$$

Here we are using the fact that  $|w_\nu(x, y)| \leq C_T$  for  $|x|, |y| \leq T$ .

If we repeat the stationary phase argument that was used to prove (2.29), we see that the left side of (2.34) is dominated by a fixed constant times

$$\lambda^{\frac{1}{2}} w_0(\tilde{x}, \alpha_j(\tilde{y})) (d_{\tilde{g}}(\tilde{x}, \alpha_j(\tilde{y})))^{-\frac{1}{2}},$$

and, consequently, we would have (2.29) if

$$(2.36) \quad w_0(\tilde{x}, \alpha_j(\tilde{y})) (d_{\tilde{g}}(\tilde{x}, \alpha_j(\tilde{y})))^{-\frac{1}{2}} \leq C \left( \max\{d_g(\tilde{x}, \alpha_j(\tilde{y})), e^{\kappa d_g(\tilde{x}, \alpha_j(\tilde{y}))}\} \right)^{-\frac{1}{2}}$$

assuming, as above, that the curvature of  $M$  is  $\leq -\kappa^2$ ,  $\kappa \geq 0$ . The last inequality comes from the fact that in geodesic normal coordinates about  $x$ , we have

$$w_0(x, y) = (\det g_{ij}(y))^{-\frac{1}{4}},$$

(see [1], [8] or §2.4 in [20]). If  $y$  has geodesic polar coordinates  $(t, \theta)$  about  $x$ , then  $t = d_{\tilde{g}}(x, y)$ , and if  $\mathcal{A}(t, \theta)$  is as in (2.12), we conclude that  $w_0(x, y) = \sqrt{t/\mathcal{A}(t, \theta)}$ , and therefore (2.36) follows from Günther's comparison estimate (2.14) if  $-\kappa^2 < 0$  and (2.13) if  $\kappa = 0$ .

The second estimate (2.35) is elementary and left for the reader, who can check that the terms are actually  $O(\lambda^{\frac{1}{2}-\nu})$ . (This is also just a special case of Lemma 3.5.3 in [20].) This completes the proof of (2.33), and, hence, that of Theorem 1.1.  $\square$

### 3. Concluding remarks.

It is straightforward to see that the proof of Theorem 1.1 shows that one can strengthen our main estimate (1.1) in a natural way. Specifically, if  $\gamma_0$  is a periodic geodesic of length  $\ell(\gamma_0)$  and if we define the  $\delta$ -tube about  $\gamma$  to be

$$\mathcal{T}_\delta(\gamma_0) = \{y \in M : \text{dist}_g(y, \gamma_0) < \delta\},$$

with  $\delta > 0$  fixed, then there is a uniform constant  $C_\delta$  so that whenever  $\varepsilon > 0$  we have for large  $\lambda$

$$(3.1) \quad \frac{1}{\ell(\gamma_0)} \int_{\gamma_0} |e_\lambda|^2 ds \leq \varepsilon \lambda^{\frac{1}{2}} \|e_\lambda\|_{L^2(\mathcal{T}_\delta(\gamma_0))}^2 + C_{\gamma_0, \delta, \varepsilon} \|e_\lambda\|_{L^2(M)}^2.$$

Thus, (1.1) essentially lifts to the cylinder  $\mathbb{R}^2/G_{\gamma_0}$ , with, as above,  $G_{\gamma_0}$ , being the stabilizer group for the lift of  $\gamma_0$  to the universal cover  $(\mathbb{R}^2, \tilde{g})$ .

To prove this, we as before write  $I = B_\varepsilon + b_\varepsilon$ , with  $b_\varepsilon(x, \xi)$  equal to one near  $T^*\gamma_0$  but supported in a small conic neighborhood of this set. Since the analog of (2.16) is valid, i.e.,

$$(3.2) \quad \int_{\gamma_0} \left| \frac{1}{\pi T} \int_{-T/4}^{T/4} \hat{\chi}(t/T) e^{-i\lambda t} B_\varepsilon \circ \cos(t\sqrt{-\Delta_g}) f dt \right|^2 ds \leq C'_{T, \varepsilon, \gamma_0} \|f\|_{L^2(M)}^2,$$

it suffices to show that

$$\frac{1}{\ell(\gamma_0)} \int_{\gamma_0} \left| \frac{1}{T} \int_{-T/4}^{T/4} \hat{\chi}(t/T) e^{-i\lambda t} b_\varepsilon \circ \cos(t\sqrt{-\Delta_g}) e_\lambda dt \right|^2 ds$$

is dominated by the right side of (3.1).

If  $K_\varepsilon(x, s)$ ,  $x \in M$ ,  $s \in \gamma_0$  denotes the kernel of this operator then, if  $\delta > 0$  and  $T$  are fixed, it follows that

$$(3.3) \quad |K_\varepsilon(x, s)| \leq C_{\gamma_0, T, \delta}, \quad x \notin \mathcal{T}_\delta(\gamma_0),$$

provided that  $b_\varepsilon$  is supported in a sufficiently small conic neighborhood of  $T^*\gamma_0$ . This is a simple consequence of the fact that when  $b_\varepsilon$  is as above, by (2.17),  $(b_\varepsilon \circ \cos t\sqrt{-\Delta_g})(x, s)$  is smooth when  $x \notin \mathcal{T}_\delta(\gamma_0)$ ,  $s \in \gamma_0$  and  $|t| \leq T$ . Since (2.21) is valid, we conclude that there is a uniform constant  $C$  so that for large  $\lambda$  we have

$$(3.4) \quad \frac{1}{\ell(\gamma_0)} \int_{\gamma_0} \left| \frac{1}{T} \int_{-T/4}^{T/4} \hat{\chi}(t/T) e^{-i\lambda t} b_\varepsilon \circ \cos(t\sqrt{-\Delta_g}) e_\lambda dt \right|^2 ds \leq CT^{-1} \lambda^{\frac{1}{2}} \|e_\lambda\|_{L^2(\mathcal{T}_\delta(\gamma_0))}^2 + C_{T, \delta, \gamma_0} \|e_\lambda\|_{L^2(M)}^2,$$

which along with (3.2) gives us (3.1). This is because we can dominate the quantity in (3.4) by the sum of the corresponding expression where  $e_\lambda$  is replaced by  $\mathbf{1}_{\mathcal{T}_\delta(\gamma_0)} e_\lambda$  and  $\mathbf{1}_{\mathcal{T}_\delta^c(\gamma_0)} e_\lambda$  and use (2.21) and our earlier arguments to show that the first of these terms is dominated by the first term in the right side of (3.4) if  $\lambda$  is large, while the second such term is dominated by last term in the right side of (3.4) on account of (3.3).

We would also like to point out that it seems likely that one should be able to take the parameter  $T$  in the proof of either (1.1) or (3.1) to be a function of  $\lambda$ . This would also require that the parameter  $\varepsilon$  to also be a function of  $\lambda$ , and thus the argument would be more involved. It would not be surprising if, as in Bérard [1] or Hassell and Tacey [9], one could take  $T$  to be  $\approx \log \lambda$ , in which case the  $L^2$ -restriction bounds in Theorem 1.1 and the  $L^4$ -estimates in Corollary 1.2 could also be improved to be  $O(\lambda^{\frac{1}{4}}(\log \lambda)^{-\delta_1})$  and  $O(\lambda^{\frac{1}{8}}(\log \lambda)^{-\delta_2})$ , respectively, for some  $\delta_j > 0$ . It is doubtful that these bounds would be optimal, though—indeed if a difficult conjecture of Rudnick and Sarnak [14] were valid, both would be  $O(\lambda^\varepsilon)$  for any  $\varepsilon > 0$ . One of the main technical issues in carrying out the analysis when  $T$  depends on  $\lambda$  would be to determine the analog of (2.15) in this case. One would also have to take into account more carefully size estimates for the coefficients  $w_\nu$ ,  $\nu > 0$ , in the Hadamard parametrix, but Bérard [1] carried out an analysis of these that would seem to be sufficient if  $T \approx \log \lambda$ . On the other hand, we have argued here that the  $w_0$  coefficient is very well behaved, and so perhaps there could be further grounds for improvement.

## REFERENCES

- [1] P. H. Bérard, *On the wave equation on a compact manifold without conjugate points*, Math. Z. **155** (1977), 249–276.
- [2] J. Bourgain, *Geodesic restrictions and  $L^p$ -estimates for eigenfunctions of Riemannian surfaces*, Linear and complex analysis, 27–35, Amer. Math. Soc. Transl. Ser. 2, 226, Amer. Math. Soc., Providence, RI, 2009.
- [3] N. Burq, P. Gérard and N. Tzvetkov: *Restriction of the Laplace-Beltrami eigenfunctions to submanifolds*, Duke Math. J. **138** (2007), 445–486.
- [4] E. Cartan, *Leçons sur la Géométrie de Espaces de Riemann* (2nd ed.) Paris, Gauthier-Villars, 1946.
- [5] J. J. Duistermaat and V. W. Guillemin, *The spectrum of positive elliptic operators and periodic bicharacteristics*, Invent. Math. **29** (1975), 39–79.
- [6] P. Günther, *Einige Sätze über das Volumenelement eines Riemannschen Raumes*, Publ. Math. Debrecen **7** (1960), 78–93.
- [7] J. Hadamard, *Les surfaces à courbures opposées et leurs géodésiques*. J. Math. Pures Appl. **4** (1898), 27–73.
- [8] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations* Dover Publications, New York, 1953.
- [9] A. Hassell and M. Tacey, personal communication.
- [10] L. Hörmander: *Fourier integral operators. I*, Acta Math. **127** (1971), 79–183.
- [11] Hörmander, *The analysis of linear partial differential operators. III. Pseudodifferential operators*, Springer-Verlag, Berlin, 1985.
- [12] P. Kröger, *An extension of Günther's volume comparison theorem*. Math. Ann. **329** (2004), 593–596.
- [13] A. Reznikov, *Norms of geodesic restrictions for eigenfunctions on hyperbolic surfaces and representation theory*, arXiv:math.AP/0403437.
- [14] Z. Rudnick and P. Sarnak: *The behaviour of eigenstates of arithmetic hyperbolic manifolds*, Comm. Math. Phys. **161** (1994), 195–213.
- [15] H. Smith and C. D. Sogge, *On the  $L^p$  norm of spectral clusters for compact manifolds with boundary*, Acta Math. **198** (2007), 107–153.
- [16] C. D. Sogge, *Oscillatory integrals and spherical harmonics*, Duke Math. J. **53** (1986), 43–65.
- [17] C. D. Sogge, *Concerning the  $L^p$  norm of spectral clusters for second-order elliptic operators on compact manifolds*, J. Funct. Anal. **77** (1988), 123–138.
- [18] C. D. Sogge, *Fourier integrals in classical analysis*, Cambridge Tracts in Math., Cambridge Univ. Press, Cambridge, 1993.
- [19] C. D. Sogge, *Keakeya-Nikodym averages and  $L^p$ -norms of eigenfunctions*, (arXiv:0907.4827) to appear Tohoku Math. J (centennial edition).
- [20] C. D. Sogge, *Hangzhou lectures on eigenfunctions of the Laplacian*, (in preparation), [www.mathematics.jhu.edu/sogge/zju](http://www.mathematics.jhu.edu/sogge/zju).
- [21] C. D. Sogge, J. Toth and S. Zelditch, *About the blowup of quasimodes, on Riemannian manifolds*, (arXiv:0908.0688) to appear, J. Geom. Anal.
- [22] C. D. Sogge and S. Zelditch, *Riemannian manifolds with maximal eigenfunction growth*, Duke Math. J. **114** (2002), 387–437.
- [23] C. D. Sogge and S. Zelditch, *Concerning the  $L^4$  norms of typical eigenfunctions on compact surfaces*, (arXiv:1011.0215) to appear.
- [24] F. Spinu, *The  $L^4$  norm of the Eisenstein series*, Thesis, Princeton University 2003, [www.math.jhu.edu/~fspinu](http://www.math.jhu.edu/~fspinu).
- [25] D. Tataru, *On the regularity of boundary traces for the wave equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **26** (1998), 185–206.
- [26] H. von Mangolt, *Über diejenigen Punkte auf positiv gekrümmten Flächen, welche die Eigenschaft haben, dass die von ihnen ausgehenden geodätischen Linien nie aufhören, kürzeste Linien zu sein*, J. Reine Angew. Math. **91** (1881), 23–52.
- [27] A. Zygmund, *On Fourier coefficients and transforms of two variables*, Studia Math. **50** (1974), 189–201.

JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD

NORTHWESTERN UNIVERSITY, EVANSTON, IL